16.3. The fundamental theorem for line integrals

Recall: If $f(x)$ is continuous on $[a,b]$ with antiderivative $F(x)$,

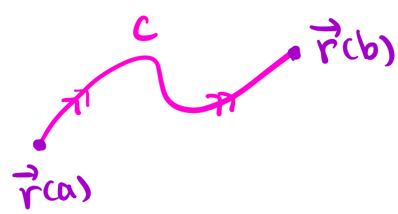
then $\int_a^b f(x)\,dx = F(b) - F(a)$.

<u>Def</u> A vector field $\vec{F}$ is <u>conservative</u> if it is the gradient of some scalar function f, called a <u>potential function</u>.

<u>Thm</u> (The fundamental theorem for line integrals)

Let C be a curve parametrized by $\vec{r}(t)$ on $a \le t \le b$.

If $\vec{F}$ is a conservative vector field with a potential function f, then



$$\int_C \vec{F}\cdot d\vec{r} = \underbrace{f(\vec{r}(b))}_{\text{terminal value}} - \underbrace{f(\vec{r}(a))}_{\text{initial value}}$$

(work done by $\vec{F}$ = difference in potential)

In particular, if C is a loop then $\int_C \vec{F}\cdot d\vec{r} = 0$.

(no work done along a loop $\Rightarrow$ energy conserved)

<u>Note</u> (1) A loop is also called a <u>closed</u> curve

* This notion is completely unrelated to the notion of closed domain

(2) Not all vector fields are conservative.

Q. How do we know if a vector field is conservative?

Remark If $\vec{F} = (P, Q)$ is conservative with a potential function f, then $P = f_x$ and $Q = f_y$

$$\Rightarrow \frac{\partial P}{\partial y} = f_{xy} = f_{yx} = \frac{\partial Q}{\partial x}.$$

This turns out to be the only condition to check in good situations.

Def A domain D in $\mathbb{R}^2$ is <u>simply connected</u> if it is connected without any holes.

e.g.

simply connected　　　not simply connected

☆ **Thm** Let D be an open and <u>simply connected</u> domain in $\mathbb{R}^2$.

A vector field $\vec{F} = (P, Q)$ is conservative on D

if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Thm A vector field $\vec{F} = (P, Q, R)$ is conservative on $\mathbb{R}^3$

if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$, $\dfrac{\partial R}{\partial x} = \dfrac{\partial P}{\partial z}$.

To be revisited in 16.5.

Ex Consider the force field $\vec{F}(x,y) = (2x-y,\ 4y-x)$.

(1) Is $\vec{F}$ conservative on $\mathbb{R}^2$?

Sol $\mathbb{R}^2$ is open and simply connected.

$$P = 2x-y,\ Q = 4y-x \Rightarrow \frac{\partial P}{\partial y} = -1,\ \frac{\partial Q}{\partial x} = -1$$

$\Rightarrow \vec{F}$ is $\boxed{\text{conservative on } \mathbb{R}^2}$

(2) Find a potential function f of $\vec{F}$.

Sol We want $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$

$$\int P\,dx = \int 2x-y\,dx = x^2 - xy$$

$$\int Q\,dy = \int 4y-x\,dy = 2y^2 - xy$$

Idea Collect all terms without duplicates

$\Rightarrow f(x,y) = \boxed{x^2 - xy + 2y^2}$.

(3) Find the work done by $\vec{F}$ along a curve C_1 from $(1,0)$ to $(2,1)$

Sol $\displaystyle\int_{C_1} \vec{F}\cdot d\vec{r} \underset{\substack{\uparrow \\ \text{Fund. thm}}}{=} f(2,1) - f(1,0) = 4 - 1 = \boxed{3}$

(4) Find a curve C_2 with $\displaystyle\int_{C_2} \vec{F}\cdot d\vec{r} = 2$.

Sol Let A and B be the start and end of C_2.

$$\int_{C_2} \vec{F}\cdot d\vec{r} = f(B) - f(A) = 2$$

Take $A = (0,0)$, $B = (0,1) \Rightarrow f(A) = 0$, $f(B) = 2$.

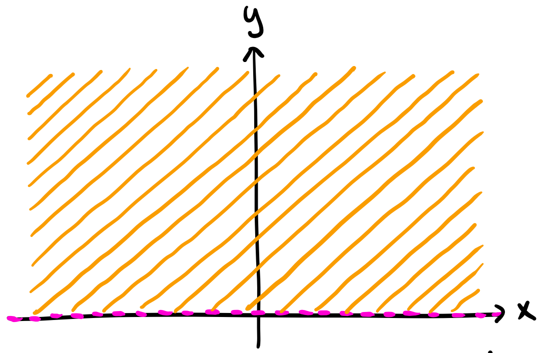
$\Rightarrow C_2$ is $\boxed{\text{a curve from } (0,0) \text{ to } (0,1)}$

Note There are infinitely many possible answers for C_2.

Ex Consider the vortex field $\vec{V}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

(1) Is $\vec{V}$ conservative on the domain $y>0$?

Sol



The domain is open and simply connected.

$$P = -\frac{y}{x^2+y^2}, \quad Q = \frac{x}{x^2+y^2}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = -\frac{1\cdot(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

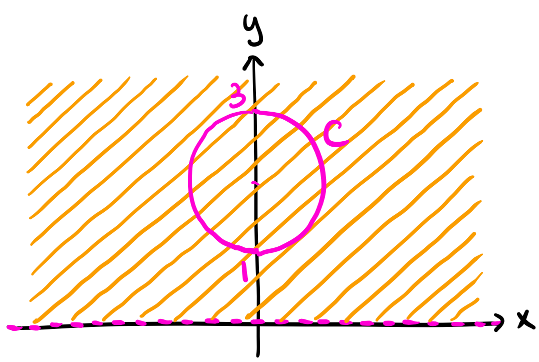
$\Rightarrow$ $\vec{V}$ is $\boxed{\text{conservative on the domain } y>0}$.

Note In fact, $\vec{V}$ has a potential function given by

$$v(x,y) = \arctan\left(-\frac{x}{y}\right)$$

(2) Find $\int_C \vec{V}\cdot d\vec{r}$ where C is given by $x^2+(y-2)^2=1$.

Sol



C is the circle of radius 1 and center (0, 2)

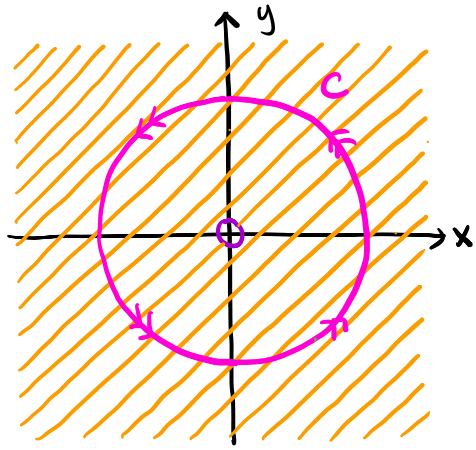
$\Rightarrow$ C lies in the domain $y>0$.

$$\Rightarrow \int_C \vec{V}\cdot d\vec{r} \underset{\uparrow}{=} \boxed{0}$$

Fund. Thm.

(3) Is $\vec{V}$ conservative on the domain $(x,y) \neq (0,0)$?

Sol



The domain is not simply connected.

(A hole at the origin)

Take C to be a circle centered at $(0,0)$ with counterclockwise orientation.

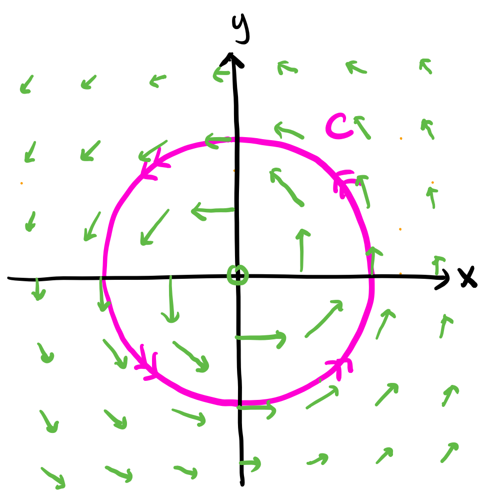
If $\vec{V}$ is conservative, then $\int_C \vec{V} \cdot d\vec{r} = 0$ ← Fund. thm.

However, we know $\int_C \vec{V} \cdot d\vec{r} = 2\pi$ ← Lecture 31

$\Rightarrow \vec{V}$ is $\boxed{\text{not conservative on the domain } (x,y) \neq (0,0)}$

Note (1) The potential function $V(x,y) = \arctan\left(-\frac{x}{y}\right)$ for the domain $y > 0$ does not work here because it is undefined on the x-axis.

(2) Intuitively, $\vec{V}$ is not conservative because of its circular (or spiral) flow.



$\vec{V}$ and C move in the same direction

$\Rightarrow$ Work done by $\vec{V}$ along C is positive

$\Rightarrow \int_C \vec{V} \cdot d\vec{r} > 0$.

$\Rightarrow \vec{V}$ is not conservative.